Задание. Разложить функцию $y = f(x)$ в ряд Фурье на промежутке $[-\pi;\pi]$. Простроить график сумм ряда Фурье.

1.3 $y = \begin{cases} \frac{x}{7}+1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

Решение.

Разложим функцию в ряд Фурье на отрезке$[-\pi;\pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

где коэффициенты вычисляются следующим образом:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}(\int_{-\pi}^{0}(\frac{x}{7}+1)dx + 1\cdot\int_{0}^{\pi}dx) = \frac{1}{\pi}\cdot((\frac{x^2}{14}+x)\Big|_{-\pi}^{0} + x\Big|_0^{\pi}) =$$

$$= \frac{1}{\pi}\cdot(\frac{\pi^2}{14}+\pi+\pi) = \frac{\pi+14}{14};$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx =$$

$$= \frac{1}{\pi}(\int_{-\pi}^{0}(\frac{x}{7}+1)\cos nxdx + 1\cdot\int_{0}^{\pi}\cos nxdx) = \frac{1}{\pi}\cdot(\frac{1}{7}\int_{-\pi}^{0}x\cdot\cos nxdx + \sin nx\big|_{-\pi}^{0} + \sin nx\big|_0^{\pi}) =$$

$$= \frac{1}{7\pi}\int_{-\pi}^{0}x\cdot\cos nxdx + 0 = \frac{1}{7\pi}(\frac{1}{n}x\sin nx\big|_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nxdx) = \frac{1}{7\pi}(0 - \frac{1}{n^2}\cos nx\big|_{-\pi}^{0}) =$$

$$= -\frac{1}{7\pi n^2}(\cos 0 - \cos(-n\pi)) = \frac{1}{7\pi n^2}(1-\cos n\pi) = \frac{1-(-1)^n}{7\pi n^2} = \frac{1+(-1)^{n+1}}{7\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin xdx = \frac{1}{\pi}(\int_{-\pi}^{0}(\frac{x}{7}+1)\sin nxdx + 1\cdot\int_{0}^{\pi}\sin nxdx) =$$

$$= \frac{1}{\pi}\cdot(\frac{1}{7}\int_{-\pi}^{0}x\cdot\sin nxdx - \frac{1}{n}\cos nx\big|_{-\pi}^{0} - \frac{1}{n}\cos nx\Big|_0^{\pi}) =$$

$$= \frac{1}{\pi}\cdot(\frac{1}{7}\int_{-\pi}^{0}x\cdot\sin nxdx - \frac{1}{n}(\cos(n\pi) - \cos(-n\pi))) =$$

$$=\frac{1}{\pi}\cdot\left(-\frac{x}{7n}\cos nx\Big|_{-\pi}^{0}+\frac{1}{7n}\int_{-\pi}^{0}\cos nx dx\right)-0=$$

$$=\frac{1}{\pi}\cdot\left(0-\frac{\pi}{7n}\cos(-n\pi)+\frac{1}{7n^2}\sin nx\Big|_{-\pi}^{0}\right)=-\frac{1}{7n}(-1)^n+\frac{1}{7n^2}(\sin 0-\sin(-\pi n))=\frac{1}{7n}(-1)^{n+1}$$

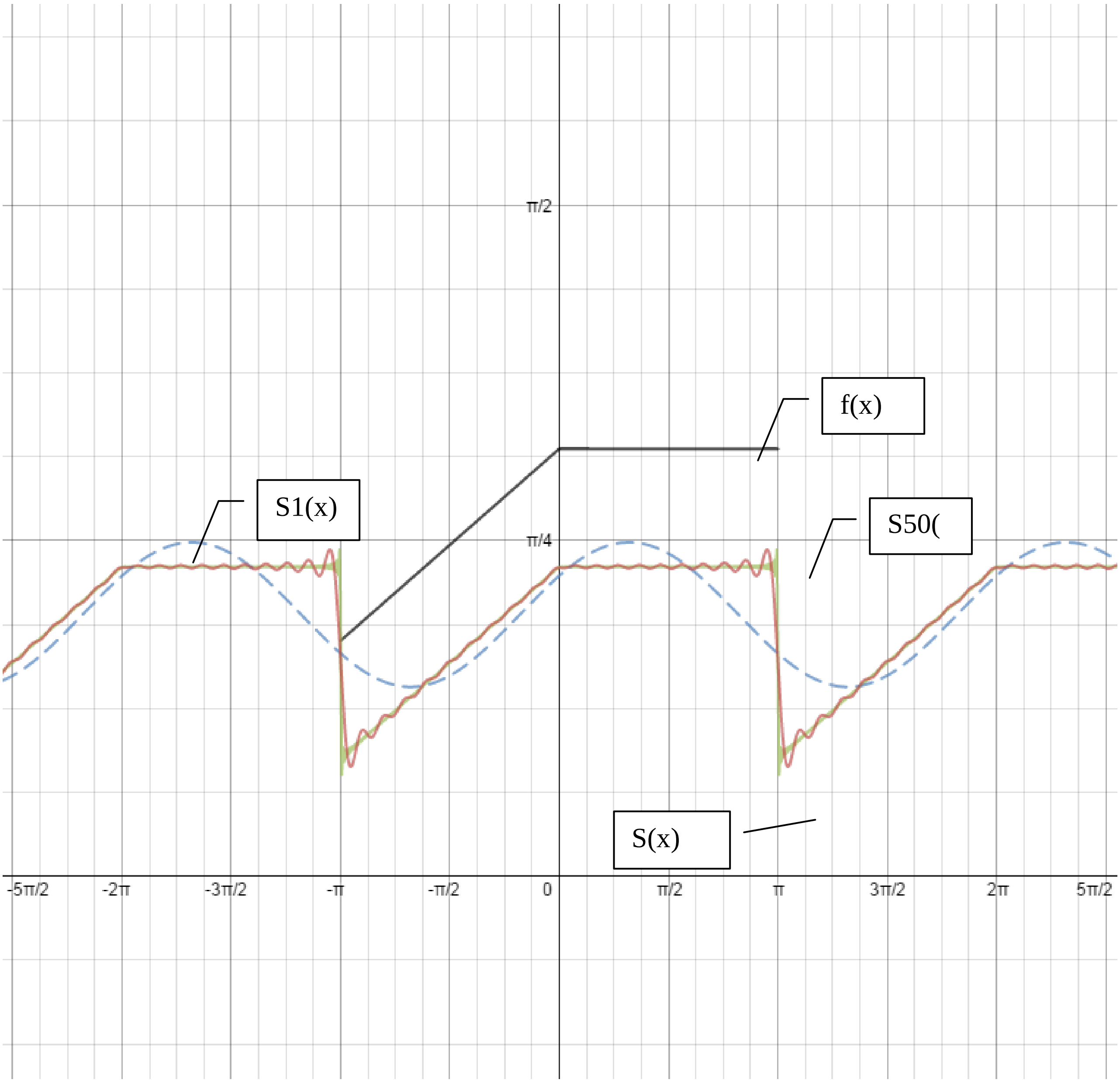
Получаем:

$$f(x)\approx\frac{\pi}{28}+\sum_{n=1}^{\infty}\left(\frac{1+(-1)^{n+1}}{7\pi n^2}\cos nx+\frac{1}{7n}(-1)^{n+1}\sin nx\right)$$

$$S(x)=\frac{\pi+14}{28}+\sum_{n=1}^{\infty}\left(\frac{1+(-1)^{n+1}}{7\pi n^2}\cos nx+\frac{1}{7n}(-1)^{n+1}\sin nx\right)$$

$$S_1(x)=\frac{\pi+14}{28}-\frac{2}{7\pi}\cos x+\frac{1}{7}\cdot\sin x$$

Графики функции и суммы ряда:


f(x)
S1(x)
S50(
S(x)
π/2
π/4
-5π/2
-2π
-3π/2
-π
-π/2
0
π/2
π
3π/2
2π
5π/2