Задача. *Разложить в ряд Фурье на* $[-\pi, \pi]$ *функцию* $f(x)=\begin{cases}-x, & x\in(-\pi, 0),\\ 3, & x\in[0, \pi].\end{cases}$

Решение. Ряд Фурье имеет вид:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right).$$

Вычислим коэффициенты ряда:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(-x)dx+\frac{1}{\pi}\int_{0}^{\pi}3dx=-\frac{1}{2\pi}x^2\Big|_{-\pi}^{0}+\frac{1}{\pi}3x\Big|_{0}^{\pi}=\frac{1}{2}\pi+3.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nxdx+\frac{1}{\pi}\int_{0}^{\pi}3\cos nxdx=$$

$$=\begin{vmatrix}u=x & du=dx\\ dv=\cos nxdx & v=\frac{1}{n}\sin nx\end{vmatrix}=$$

$$=-\frac{1}{\pi}\frac{1}{n}x\sin nx\Big|_{-\pi}^{0}+\frac{1}{\pi}\frac{1}{n}\int_{-\pi}^{0}\sin nxdx+\frac{3}{n\pi}\sin(nx)\Big|_{0}^{\pi}=$$

$$=-\frac{1}{\pi}\frac{1}{n^2}\cos nx\Big|_{-\pi}^{0}=\frac{(-1)^n-1}{\pi n^2}.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nxdx+3\frac{1}{\pi}\int_{0}^{\pi}\sin nxdx=$$

$$=\begin{vmatrix}u=x & du=dx\\ dv=\sin nxdx & v=-\frac{1}{n}\cos nx\end{vmatrix}=$$

$$=\frac{1}{\pi}\frac{1}{n}x\cos nx\Big|_{-\pi}^{0}-\frac{1}{\pi}\frac{1}{n}\int_{-\pi}^{0}\cos nxdx-\frac{3}{n\pi}\cos(nx)\Big|_{0}^{\pi}=$$

$$=\frac{\pi(-1)^n}{n\pi}-\frac{1}{\pi}\frac{1}{n^2}\sin nx\Big|_{-\pi}^{0}-3\frac{(-1)^n-1}{n\pi}=\frac{3-3(-1)^n+\pi(-1)^n}{\pi n}.$$

Получаем ряд Фурье**:**

$$f(x)=\frac{\pi+6}{4}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{\pi n^2}\cos nx+\frac{3-3(-1)^n+\pi(-1)^n}{\pi n}\sin nx\right)$$

Ответ: $f(x)=\frac{\pi+6}{4}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{\pi n^2}\cos nx+\frac{3-3(-1)^n+\pi(-1)^n}{\pi n}\sin nx\right)$